BOOLEAN [0,1]-VALUED CONTINUOUS OPERATORS

JINSEI YAMAGUCHI

Dept. of Information Science, KANAGAWA UNIVERSITY

2946 Tsuchiya, Hiratsuka, Kanagawa 259-12 JAPAN

ABSTRACT: We prove the following claim.

Theorem: There are continuous operators $*$, $\perp$, $\#$ on [0,1] such that $([0,1],*,\perp,\#)$ approximate a Boolean algebraic structure on [0,1] for an arbitrary preciseness. □

In other words, for any $\varepsilon > 0$, we can continuous functionally define a Boolean algebraic structure on [0,1] modulo ε. Here, the meaning of " modulo ε " is the following. We can take points $\{r_1,\cdots,r_{2^n}\}$ in [0,1] such that

$$\max\{(r_{i+1}-r_i) \mid 0 \leqq i \leqq 2^n-1\} < \varepsilon ,$$

so that $(\{r_1,\cdots,r_{2^n}\},*,\perp,\#)$ becomes a Boolean algebra.

KEY WORDS: Boolean algebra, logic, truth-value, uncertainty, probability

C.R.CATEGORIES: F.3, F.4, I.2

§1. INTRODUCTION

Consider the case that we meet the issue of the [0,1]-valued assignment to a set of propositions (,or more generally, data, information and so on). Such cases are often encountered almost everywhere in AI, say, in the fields of ES, Data & Knowledge Base, Problem Solving, Searching, Machine Learning, Pattern Recognition, Natural Language and/or Image Understanding, Fuzzy Theory, GA, Multi-Agent, CAI etc. In each field, the contents of the assignment may vary from a truth-value to a certainty factor or a probability or a vagueness or a fuzziness etc. In this paper, in order to argue our claim in the abstract setting, we do not specify the content of the assignment. What we are concerned with is whether it is compositional w.r.t. connectives $\wedge$, $\vee$, $\neg$ or not. To be more precise, we are solely interested in the following problem.

Let K be the set of propositions (data, information) and let τ be an arbitrary [0,1]-valued assignment technique to K. In general, there exist many kind of compound propositions generated from $\wedge$, $\vee$, $\neg$ in K. The point is whether the following claim is true or not.

Claim: there are ***continuous*** operators $*$, $\perp$, $\#$ on [0,1] such that, for any a, $b \in K$,

(i) $\tau(a \wedge b) = \tau(a) * \tau(b)$

(ii) $\tau(a \vee b) = \tau(a) \perp \tau(b)$

(iii) $\tau(\neg a) = \# \tau(a)$. ⋯(1)

If there are not such operators $*$, $\perp$, $\#$ on [0,1], the calculation of [0,1]-value for a complex proposition becomes ad-hoc and the semantics of the connectives $\wedge$, $\vee$, $\neg$ become computationally incoherent. So, it is natural for us to expect that there are such operations.

On the other hand, (1) is by no means obvious. To see this, we need to distinguish the algebraic structure of $(K, \wedge, \vee, \neg)$ and the [0,1]-valued semantics of $\wedge, \vee, \neg$ defined by $*$, $\perp$, $\#$. Connectives $\wedge, \vee, \neg$ on K can be defined before we select the corresponding operators $*$, $\perp$, $\#$ on [0,1]. They may be classical or non-classical or even non-logical in the rigid sense.

In this situation, suppose $(K, \wedge, \vee, \neg)$ forms a Boolean algebra. The conventional recognition about the [0,1]-valued assignment has been the following:

「As far as Boolean structure concerns, there is no compositional assignment τ such that (1) holds, except the trivial case of τ being {0,1}-valued.」 ⋯(2)

For example, [2] and [3] claim this result. One of the purposes of this paper is to show that the above claim (2) is false.

§ 2. OBJECTION

In this section, we point out how the conventional proofs of the above statement (2) are wrong or misunderstood. In [2], the authors prove the following statement, which is essentially the same as (2).

「 **Proposition**. Let P be a finite Boolean algebra of propositions and let τ be a truth-assignment function P$\to$[0,1], supposedly truth-functional via continuous connectives. Then, $(\forall p \in \mathrm{P})(\tau(p) \in \{0,1\})$. Moreover, τ is an interpretation in the sense of propositional calculus, i.e.,

$\tau(p)=1 \Leftrightarrow \tau(\neg p)=0$. 」 ([2], p 295)

However, the proof of this proposition is wrong. The wrong line is

" * is a continuous monotone semigroup of [0,1] called a triangular conorm. " ([2], p 296)

where, * means $\perp$ in our terminology.

In order to obtain this wrong result, they refer to [1] Dubois and Prade. The related section in [1] is titled as " Ⅲ SET FUNCTIONS BASED ON A TRIANGULAR CONORM: 2.Arrival of Conorms " pp47-48. There, they deduce that

" As a result, * should be non-decreasing in each place; particularly $1*1=1$. It is then obvious that in (11), the combination operator cannot be but a triangular conorm. " ⋯(4)

,where (11) means

" if $A \cap B = \emptyset$, then $g(A \cup B) = g(A)*g(B)$ ".

Unfortunately, the claim (4) is false. The wrong statement in (4) is " * should be non-decreasing in each place ". There, to say * being non-decreasing, they apply (11) to the argument that

"Let (A,B) and (C,D) be two pairs of disjoint subsets of X such that $A \subseteq C$, $B \subseteq D$ (hence $g(A) \leqq g(C)$ and $g(B) \leqq g(D)$), then
$g(A \cup B) = g(A)*g(B) \leqq g(C \cup D) = g(C)*g(D)$." ([1] p 48)

However, from this argument, we can not deduce that * is non-decreasing in general, because the above says nothing about arbitrary (not disjoint) subsets of X such that $A \subseteq C$, $B \subseteq D$!

§3. MAIN THEOREM

In the previous section, we directly check how the conventional proof of the statement (2) is wrong. Contrastingly, in this section, we prove a theorem which conflicts with (2). Thus, we again show that (2) is false. Before proving our main theorem, let's state the following crucial fact.

Lemma 3-1. 《Boolean Projection》
Let $0 = r_1 < \cdots < r_{2^n} = 1$ be 2^n different real numbers in [0,1], where $n \geqq 1$ is arbitrary. Then, we can define a Boolean algebraic structure $(\{r_1, \cdots, r_{2^n}\}, \wedge_*, \vee_\perp, \neg_\#)$ so that

$$(\forall r_i, r_j \in \{r_1, \cdots, r_{2^n}\})(r_i \ll_{(\wedge,\vee,\neg)} r_j \rightarrow r_i \leqq r_j)$$

where $\ll_{(\wedge,\vee,\neg)}$ is the lattice-ordering in the sense of Boolean algebra and $\leqq$ is the real number ordering.

Proof: See Lemma 2-1 in [5]. □

Using this lemma, we can prove our main theorem.

Theorem 3-2. Let K be the set of compound propositions (data, information) generated from the connectives $\wedge, \vee, \neg$ such that $(K, \wedge, \vee, \neg)$ forms a Boolean algebra, either finite or infinite. Then, there are [0,1]-valued assignment $\tau : K \to [0,1]$ and continuous operators $*$, $\perp$, $\#$ on [0,1] such that, for any a, b$\in K$, the following relations hold simultaneously.

(i) $\tau(a \wedge b) = \tau(a) * \tau(b)$

(ii) $\tau(a \vee b) = \tau(a) \perp \tau(b)$

(iii) $\tau(\neg a) = \# \tau(a)$

, where τ is not trivial, i.e., τ is not {0,1}-valued.

Proof: Let $0 = r_1 < \cdots < r_{2^n} = 1$ be arbitrary where $n \geqq 2$. Let $A \subset K$ be the set of all ground atoms and let $\tau : A \to [0,1]$ be such that

$$(\forall a \in A)(\tau(a) \in \{r_1, \cdots, r_{2^n}\})$$

Then, by using the above lemma 3-1, we can define operators $\wedge_*$, $\vee_\perp, \neg_\#$ over $\{r_1, \cdots, r_{2^n}\}$ so that τ is extended to all K satisfying

(i) $\tau(a \wedge b) = \tau(a) \wedge_* \tau(b)$

(ii) $\tau(a \vee b) = \tau(a) \vee_\perp \tau(b)$

(iii) $\tau(\neg a) = \neg_\# \tau(a)$.

So, all we need is to extend $\wedge_*$, $\vee_\perp, \neg_\#$ to the corresponding continuous operators $*$, $\perp$, $\#$ on [0,1]. The method is not so difficult. For example:

(Ⅰ) The case of $*$

Let r_i and r_j be two arbitrary elements in $\{r_1, \cdots, r_{2^n}\}$, where $1 \leqq i, j \leqq 2^n - 1$. Consider four points

$$(r_i, r_j, *(r_i, r_j)), \quad (r_i, r_{j+1}, *(r_i, r_{j+1})), \quad (r_{i+1}, r_{j+1}, *(r_{i+1}, r_{j+1})), \quad (r_{i+1}, r_j, *(r_{i+1}, r_j))$$

in $[0,1] \times [0,1] \times [0,1]$.

Using these four points, we can consider four line segments L_1, L_2, L_3, L_4 generated by two points

L_1 : $\{(r_i, r_j, *(r_i,r_j)), (r_i, r_{j+1}, *(r_i,r_{j+1}))\}$

L_2 : $\{(r_i, r_{j+1}, *(r_i,r_{j+1})), (r_{i+1}, r_{j+1}, *(r_{i+1},r_{j+1}))\}$

L_3 : $\{(r_{i+1}, r_{j+1}, *(r_{i+1},r_{j+1})), (r_{i+1}, r_j, *(r_{i+1},r_j))\}$

L_4 : $\{(r_{i+1}, r_j, *(r_{i+1},r_j)), (r_i, r_j, *(r_i,r_j))\}$

respectively.

Now, we can take a continuous curved surface H_{ij} in $[0,1]\times[0,1]\times[0,1]$ surrounded by L_1, L_2, L_3, L_4 . The simplest case may be the following.

Consider the 5th line segment L_5 generated by two points

L_5: $\{(r_i, r_j, *(r_i,r_j)), (r_{i+1}, r_{j+1}, *(r_{i+1},r_{j+1}))\}$.

Then, we obtain two triangular regions R_1 and R_2 such that

R_1 : (L_1, L_2, L_5)

R_2 : (L_3, L_4, L_5).

Since R_1 and R_2 are stapled by the line segment L_5 , we can choose (R_1+R_2) as a candidate of H_{ij}.

(Here, note the fact that the above defined H_{ij} has a realization algorithm for any r_i, r_j.)

Then, the final extended continuous operation $*$ on [0,1] is defined by gathering these surfaces H_{ij} for $1\leqq i,j\leqq 2^n-1$.

(II) The case of $\perp$ is similar.

(III) The case of # is easier. □

§4. COROLLARIES

As a direct consequence, we notice that a statement in [3] is wrong. To be more precise:

Corollary 4-1. The following claim is false.

「There can not exist operations ⊥ and * on [0,1], nor negation function f such that the following identities simultaneously hold for all propositions S_1, S_2, S where g stands for a [0,1]-valued function which intends to estimate uncertainty.

(i) $g(\text{not } S) = f(g(S))$

(ii) $g(S_1 \wedge S_2) = g(S_1) * g(S_2)$

(iii) $g(S_1 \vee S_2) = g(S_1) \perp g(S_2)$

More precisely, (i)-(iii) entail that

$(\forall S)(\ g(S)=0 \text{ or } g(S)=1\ )$, i.e.

we are in the deterministic case where a statement is either true or false. 」 ([3] p212)

Proof: Direct consequence of the above Theorem 3-2. □

By the way, what we have done in the proof of Theorem 3-2 is that,

〈1〉 firstly, choose a subset Ｔ of [0,1],

〈2〉 secondly, define continuous operations ＊, ⊥, ＃ on [0,1] so that they are closed on Ｔ.

As the consequence, the range of τ does not cover the total [0,1]. This means that the claim stated in Theorem 3-2 does not contradict the famous statement

" [0,1] can not be equipped with a Boolean algebraic structure " …(5)

in this field. Concerning this aspect, as another direct consequence of Theorem 3-2, we obtain our main result.

Corollary 4-2. For any $\varepsilon > 0$, we can continuous functionally define a Boolean algebraic structure on [0,1] modulo ε. □

Here, the meaning of the expression of “ modulo ε ” is the following. In the proof of Theorem 3-2, we employ the extension strategy of $*$, $\perp$, # from the domain $\{r_1,\cdots,r_{2^n}\}$ to all [0,1]. As the result, we obtain the block-wise Boolean-valued structure over [0,1], in the sense that each $[r_i,r_{i+1})$ forms an element of a Boolean algebra. (The topmost element is exceptionally $[1,1]=1$) Thus, “ modulo ε ” means that

$$\max\{(r_{i+1}-r_i) \mid 0\leqq i\leqq 2^n-1\}<\varepsilon .$$

Here, remember that it often happens that [0,1] is quantified to the representatives $\{c_1,\cdots,c_m\}$ in a practical phase. So, by taking $n\geqq 2$ such that $2^n\geqq m$, Theorem 3-2 inform an crucial fact in this field.

§5. INVESTIGATION

So far, we have shown how the conventional statement (2) has been mistakenly believed to be true. Some researchers have the intention to try to prove the result (2), but in vain! What does this mean in the history of AI or logic ? This clearly means that the issue (1) is neither trivial nor obvious. But then, why such a wrong legend as (2) was born in this area ? To investigate the reason would contribute not only to the field of philosophical logic but also to AI, we hope. So, in this section, let's devote ourselves to this task.

By taking [0,1] as the target domain, we can use the computational property of [0,1] to define operators $*$, $\perp$, #. In this situation, we believe that the seeds of the mistake were sown by the confusion of the lattice ordering $\ll_{(\wedge,\vee,\neg)}$ of the logical structure $(K,\wedge,\vee,\neg)$ with the real number ordering $\leqq$ of [0,1]. Some researchers might think that, as far as [0,1] concerns, $\ll_{(\wedge,\vee,\neg)}$ should be identical with $\leqq$. However, this thought is not true at all.

As a matter of fact, for the same logical structure $(K,\wedge,\vee,\neg)$, we can employ many kind of target domains, say, {0,1}, $\{1,\cdots,n\}$, N(the set of all natural numbers), R(the set of all real numbers), $[-1,1]$ etc. [0,1] is nothing but one candidate of them. Similarly, we can choose many kind of logical structures for the same target domain [0,1]. These facts clearly demonstrate that two orderings $\ll_{(\wedge,\vee,\neg)}$ and $\leqq$ are essentially different.

Of course, there are some relations between $\ll_{(\wedge,\vee,\neg)}$ and ≦. The relations arise from the fact that operators ＊, ⊥, ＃ corresponding to ∧,∨,¬ are defined by using the computational properties of [0,1], which are also connected with the real number ordering ≦.

For example, as far as the operator ＊ concerns, researchers in this field often employ the following criteria, which implicitly relates $\ll_{(\wedge,\vee,\neg)}$ with ≦.

〔1〕 ＊(0,0)＝0 and ＊(r,1)＝＊(1,r)＝r for all r∈[0,1] (Boundary Condition)

〔2〕 ＊(r,s)＝＊(s,r) for all r,s∈[0,1] (Commutativity)

〔3〕 ＊(r,＊(s,t))＝＊(＊(r,s),t) for all r,s,t∈[0,1] (Associativity)

〔4〕 ＊(s,t)≦＊(u,v) for all s≦u, t≦v∈[0,1] (Monotonicity)

This criteria for ＊ is called "T-norm", where property 〔4〕 is directly related to the ordering ≦. This notion of T-norm is expected to extend the usual notion of "∧(inf)" in the sense of lattice ordering from {0,1} to [0,1]. However, as far as the extension of the notion of ∧ concerns, the following natural property should also be considered.

〔5〕 ＊(r,r)＝r for all r∈[0,1] (Idempotentness)

Here, one interesting question from a viewpoint of philosophical logic or AI is the following.

「Which is a more natural property for the extension of ∧,〔4〕 or〔5〕?」

Why this question is so interesting? Because, we believe, this attitude toward the preference of two properties〔4〕 and〔5〕 is the frontier which distinguish the answer to our theme (2), yes or no!

To be more precise, we notice the following. The most famous examples of T-norm is the min-operator on [0,1]. This operator satisfies the property〔5〕. There is another famous example of T-norm. This is the usual product × on [0,1]. However, × does not satisfy the property〔5〕. On the other hand, the operator ＊ defined in Theorem 3-2 of this paper satisfies 〔1〕＋〔2〕＋〔3〕＋〔5〕, but it does not satisfy 〔4〕. In this sense, × and our $*$ are competitive notions w.r.t. properties 〔4〕 and 〔5〕. Here, we should not forget the fact that the claim (2) does not put the restriction on the candidates of $*$. That is, it need not to be a T-norm.

Similar argument can be applied to $\perp$, too.

§5. CONCLUSION

Everyone in this field admits that the target domain [0,1] is an extension of the truth-value domain {0,1}. At the same time, it is well-known that there is no linearly ordered Boolean algebra except the classical {0,1}-valued Boolean algebra $\mathbb{2}$. From these two results, however, we can not deduce the claim (2). The reason is that the lattice ordering $\ll_{(\wedge,\vee,\neg)}$ in the sense of logical system $(K,\wedge,\vee,\neg)$ is different from the real number ordering $\leqq$ on [0,1], though both orderings are related to each other by operators $*$, $\perp$, $\#$ on [0,1]. It is this recognition that becomes the starting point of the profound investigation of the claim (2).

One crucial step to this direction is the distinction between the claim (2) and the claim (5). The difference is solely reduced to the role of the assignment τ. Suppose an expected model for the system $(K,\wedge,\vee,\neg)$ is fixed. This means that τ is a function and so the range D of τ becomes a proper subset of [0,1] whose cardinality is at most countable, because the cardinality of K is at most countable. Here, it is obvious that conventional proofs of (2) stand on this model theoretic aspect. This can be detected from the expression

「τ is trivial」 in [4]

or

「$(\forall p\in P)(\tau(p)\in\{0,1\})$」 in [2]

or

「$(\forall S)(g(S)=0$ or $g(S)=1)$」 in [3].

In this paper, we devote ourselves to the task of refuting (2). Similar argument can be applied to other non-classical (,or more generally, non-logical) structures. During the argument, we should legitimately solve the issue that

「Firstly, a structure $(K,\wedge,\vee,\neg)$ need to be fixed. Then find operators $*$, $\perp$, $\#$ on [0,1] such that (1) hold.」

Remark that this issue is just the converse of the issue that

「Firstly, choose operators $*$, $\perp$, $\#$ on [0,1]. Then, define the structure $(K,\wedge,\vee,\neg)$ so that (1) hold.」

The key of this latter issue is not the definability of $(K,\wedge,\vee,\neg)$ (it is always definable!), but what kind of algebraic structure does $(K,\wedge,\vee,\neg)$ have syntactically.

In anyway, the importance of the question (1) in the general setting is high-lighted.

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